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LETTER TO THE EDITOR

Exact scaling functions for self-avoiding loops and branched polymers

John Cardy

Theoretical Physics, Department of Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK
and
All Souls College, Oxford, UK

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Abstract

It is shown that a recently conjectured form for the critical scaling function for planar self-avoiding polygons weighted by their perimeter and area also follows from an exact renormalization group flow into the branched polymer problem, combined with the dimensional reduction arguments of Parisi and Sourlas. The result is generalized to higher-order multicritical points, yielding exact values for all their critical exponents and exact forms for the associated scaling functions.

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In two dimensions, exact results for critical exponents, which describe a power law dependence on a single relevant variable close to a critical point, are commonplace. By contrast, very few examples are known of exact scaling functions which depend on combinations of more than one such variable. In recent years there has been considerable progress in problems involving counting various restricted classes of random self-avoiding polygons [1]. However, these are essentially one dimensional in nature, and so far no rigorous results exist for the unrestricted case. Theoretically this latter case is perhaps more interesting because, in the scaling limit, it corresponds to an isotropic field theory, the $n \to 0$ limit of the O(n) model. While much exact information is known about such critical theories in two dimensions, up to now no exact, nontrivial, scaling functions of more than one intensive thermodynamic variable (such as the equation of state) have been found.

Recently Richard *et al* [2] (hereinafter referred to as RGJ) have conjectured the exact form of such a scaling function for unrestricted self-avoiding polygons in the plane. In the ensemble in which each link of the polygon is weighted with fugacity x, this problem exhibits a critical point at some value $x = x_c$. As $x \to x_c$ from below, the mean perimeter $\langle N \rangle$, mean square radius of gyration $\langle R^2 \rangle$ and mean area $\langle A \rangle$ all diverge. Such self-avoiding loops (SALs) provide a simple model for two-dimensional vesicles [3], and in that context it is natural to

weight the ensemble according to the area of each loop, thus defining the generating function for rooted loops

$$G^{(r)}(x, g) = \sum_{N,A} p^{(r)}_{N,A} x^N \mathrm{e}^{-gA}$$

where $p^{(r)}_{N,A}$ is the number of such loops of a given perimeter and area which pass through a given link of the lattice, and g ($\equiv -\ln q$ in the notation of RGJ) is the pressure difference across the vesicle wall, in units of kT. From the point of view of critical phenomena, this ensemble has a multicritical point at $x = x_c$, $g = 0$, and in its neighbourhood one expects [3] that the singular part of G has the scaling form (in the notation of RGJ)

$$G^{(r)}_{\mathrm{sing}}(x, g) = g^{\theta} F\left((x_c - x)g^{-\phi}\right) \tag{1}$$

where θ and ϕ are related to the conventional exponents ν and α by $\theta/\phi = 1-\alpha$ and $\phi = 1/2\nu$, based on the assumptions that, at $g = 0$, the singular part of the mean number of such rooted loops behaves like $(x_c - x)^{1-\alpha}$ and the mean area $\langle A\rangle \sim \langle R^2\rangle \sim (x_c - x)^{-2\nu} \sim \langle N\rangle^{2\nu}$. In addition α is related to ν by hyperscaling: $\alpha = 2 - 2\nu$. There is ample evidence from enumeration and other methods [4] to support these assumptions and the theoretical value [5] $\nu = \frac{3}{4}$, so that $\theta = \frac{1}{3}$ and $\phi = \frac{2}{3}$.

RGJ, in analogy with similar but analytically tractable enumeration problems [1], assume that, as a function of x and $q = \mathrm{e}^{-g}$, $G^{(r)}$ satisfies some q-algebraic functional equation of finite degree. Together with the assumed values for θ and ϕ, this leads, in the limit $q \to 1$, to a Riccati equation for the scaling function $F(s)$, whose solution is

$$F(s) = b_0 \frac{\mathrm{d}}{\mathrm{d}s} \ln\ \mathrm{Ai}\,(b_1 s) \tag{2}$$

where $\mathrm{Ai}(x) \propto \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt + \mathrm{i}t^3/3}\,\mathrm{d}t$ is the Airy function, and b_0, b_1 are non-universal constants. Equation (2) determines exactly, for example, the universal moment ratios $\langle A^p\rangle/\langle A\rangle^p$ as $x \to x_c$, and RGJ produce convincing evidence, based on extensive enumerations, that these predictions are indeed correct.

In this Letter, it is pointed out that (2) also follows from a completely different argument, which invokes the physical reasoning of [3] to relate this problem to that of branched polymers, combined with the dimensional reduction arguments of Parisi and Sourlas [6], which map this latter problem to that of the Yang–Lee edge singularity in two fewer dimensions. From this point of view, the Airy integral then arises as the scaling limit of the partition function of the Yang–Lee problem in zero dimensions. Moreover, in this approach, the values of the exponents θ and ϕ emerge without any further assumptions.

From this perspective it is simple to generalize the conjecture of RGJ to higher-order multicritical points of SALs with k relevant renormalization group (RG) scaling variables v_j. These may presumably be realized by tuning to critical values many-body interactions between nearby portions of the loop. When this ensemble is, in addition, weighted by the area of the loops, the generalization of (1) to arbitrary k is

$$G^{(r)}_{\mathrm{sing}} = g^{\theta_k} F_k(v_1 g^{-y_1/2}, v_2 g^{-y_2/2}, \ldots) \tag{3}$$

where y_j is the RG eigenvalue of v_j. It will be argued that the exact values for these, at the kth-order multicritical point, are

$$y_j(k) = 2(k - j + 2)/(k + 2) \tag{4}$$

and that the exact form for F_k is given in terms of a generalized Airy integral $\int_C \mathrm{e}^{-V(\psi)/g}\mathrm{d}\psi$ where $V(\psi) = \sum_{j=1}^{k} v_j \psi^j - \psi^{k+2}$. The values given in (4) agree with those derived from a generalized Flory argument, applied to an ensemble in which the first k renormalized virial

coefficients vanish. That they should be exact in two dimensions was suggested earlier by Saleur [7] on the basis of a postulated $N = 2$ supersymmetry. Here it is seen that they follow from the mapping to a simple zero-dimensional problem. However, our results, like those of Saleur, display a paradox in that $k = 2$, the obvious candidate for the Θ-point, yields values for the exponents which disagree with those of an exactly solvable model [15] and with extensive numerical results. This is discussed in detail later.

Finally, in the generalized Airy integral it is possible to take the limit $g \to 0$, thus recovering the scaling function in the original ensemble with no area-weighting. This comes from the appropriate saddle-point of $V(\psi)$, and therefore amounts to finding the root of a polynomial. For example, for $k = 2$ it is found that $G^{(r)}_{\rm sing} = c_0 v_2^{1/2}\, \Phi(c_1 v_1 / v_2^{3/2})$, where c_0 and c_1 are non-universal constants, and the exact scaling function, for $v_2 > 0$, is

$$\Phi^{>}(s) = \left(s + (s^2-1)^{1/2}\right)^{1/3} + \left(s - (s^2-1)^{1/2}\right)^{1/3} \tag{5}$$

where the branch cuts of the fractional powers are taken to lie along the negative real axis.

We now give more details of the reasoning leading to these results, first discussing the case $k = 1$ considered by RGJ. The physical part of the argument is to regard the model with $g > 0$ as presenting a *crossover* phenomenon: the negative pressure causes the vesicles to try to minimize their area, but there is competition between this and the need to maximize the perimeter as $x \to x_{\rm c}$. Clearly for large g at fixed $x < x_{\rm c}$ the vesicles should collapse into double-walled, branched structures, but assume, as suggested by the numerical work of [3] that, at large enough distance scales, this will also happen as $x \to x_{\rm c}$ for any fixed $g > 0$, consistent with the idea that there is an RG flow from the fixed point describing SALs to that corresponding to branched structures. Thus (1) has the form of a crossover scaling function [8]. We also assume that the structures which result are in the same universality class as conventional branched polymers (lattice animals with no cycles), in which all trees with the same total length are weighted equally. The theory of crossover scaling [8] then asserts that the scaling function $F(s)$ in (1) should have a singularity of the form $(s - s^*)^{1-\alpha_{\rm BP}}$, where $\alpha_{\rm BP}$ is the entropic exponent for branched polymers. From this hypothesis various interesting results follow, for example that as $g \to 0$ the branched polymer singularity should occur at $x = x_{\rm c}(g) = x_{\rm c} + s^* g^{\phi} + \cdots$, which has been confirmed in enumeration studies [3], as well as various predictions for the g-dependence of the critical amplitudes. In general, however, the functional form of a crossover function is very difficult to calculate, since the scaling variables at the new fixed point bear a complicated relationship to the original ones, which requires following the RG flow in detail. However in this example there are considerable simplifications.

First state the problem in field-theoretic language, by writing the area of a given loop as

$$A = \iint G_{\lambda\sigma}(r_1 - r_2) J_\lambda(r_1) J_\sigma(r_2)\, {\rm d}^2 r_1\, {\rm d}^2 r_2 \tag{6}$$

where J_λ is the density of a current of unit strength flowing around the loop, and $G_{\lambda\sigma}$ is the Green function for a $U(1)$ gauge field $\mathcal{A}$. Equation (6) expresses the well known fact that, in a two-dimensional gauge theory, the expectation value of a Wilson loop obeys a strict area law. In [9] it was used to compute the mean area of SALs. The generating function $G^{(r)}$ for rooted loops is the derivative with respect to the fugacity x of

$$Z = \langle {\rm e}^{-gA} \rangle_{\rm SAL} = \left\langle {\rm e}^{-\sqrt{g}\int J_\lambda \mathcal{A}_\lambda {\rm d}^2 r} \right\rangle_{{\rm SAL},\mathcal{A}}$$

where the average is taken over SALs, each weighted by x^N, and over the gauge field, with the usual weight $\exp\left(-\frac{1}{4}\int F^{\lambda\sigma} F_{\lambda\sigma}\, {\rm d}^2 r\right)$. SALs may be mapped, in the standard way, to the $n \to 0$ limit of an O(n) model. In this case it is useful to consider complex O(n) lattice spins $\boldsymbol{s}(r)$, so that the $U(1)$ current J_λ is the lattice version of $(1/2{\rm i})(\boldsymbol{s}^* \cdot \partial_\lambda \boldsymbol{s} - {\rm c.c.})$, and the weights are $\prod_{\rm nn}(1 + x(\boldsymbol{s}^*(r) \cdot \boldsymbol{s}(r') + {\rm c.c.}))$.

The first observation is that, at $n = 0$, there are no vacuum corrections to the gauge field propagator $G_{\lambda\sigma}$ (as in the 'quenched' approximation in lattice gauge theories), so that, since $\mathcal{A}$ couples to a conserved current, the gauge coupling g is not renormalized. Its RG equation is simply $\mathrm{d}g/\mathrm{d}\ell = 2g$ to all orders, so that it flows to infinity, where the irrelevant variable g^{-2} has RG eigenvalue -2. The other simplification is that, in the limit where g is large, the total length of the branched polymer is one-half that of the perimeter of the loop, apart from corrections $\mathrm{o}(N)$. Thus the fugacity variable for the branched polymer problem, close to x_c, is *linearly* related to the original fugacity x.

Now recall the formulation of the branched polymer problem in d dimensions, given by Parisi and Sourlas [6]. This is the $n \to 0$ limit of a theory of fields ψ_a ($a = 1, \ldots, n$), weighted by e^{-S} where

$$S = \int \left(\sum_a \left(\tfrac{1}{2}(\nabla\psi_a)^2 - \sum_{p\geqslant 1} u_p \psi_a^p \right) + v \sum_{ab} \psi_a^2 \psi_b^2 \right) \mathrm{d}^d r \tag{7}$$

where u_p is the fugacity for p branches to meet at a given point, and $v > 0$ represents self-avoidance. After shifting the fields to eliminate the ψ_a^2 term, rescaling and retaining only the most relevant couplings, the action takes the form

$$S = \int \left(\sum_a (\tfrac{1}{2}(\nabla\psi_a)^2 + V(\psi_a)) + \Delta \sum_{ab} \psi_a \psi_b \right) \mathrm{d}^d r \tag{8}$$

where $V(\psi) = t\psi - \frac{1}{3}\psi^3 + \mathrm{O}(\psi^4)$. This theory is critical at some value $t \to t_\mathrm{c}+$. Parisi and Sourlas [6] argued that, at $n = 0$, (8) is equivalent to a supersymmetric theory. We follow the more direct transformation of [10]: define new combinations of the fields $\psi \equiv \frac{1}{2}(\psi_1 + (n-1)^{-1}\sum_2^n \psi_a)$, $\omega \equiv \Delta(\psi_1 - (n-1)^{-1}\sum_2^n \psi_a)$, together with $n-2$ other fields χ_a ($a = 3, \ldots, n$) which are linear combinations of $(\psi_2, \ldots, \psi_n)$ orthogonal to $\sum_2^n \psi_a$. Discarding terms higher than quadratic order in ω and the χ_a (which may be shown to be irrelevant), the action has the form, at $n = 0$,

$$S = \frac{1}{\Delta} \int \left(\omega(-\nabla^2\psi + V'(\psi)) - \omega^2 + \sum_a \chi_a(-\nabla^2 + V''(\psi))\chi_a \right) \mathrm{d}^d r. \tag{9}$$

The integral over the $n-2$ commuting fields χ_a yields $\det(-\nabla^2 + V'')^{-(n-2)/2}$ and so they may be replaced at $n = 0$ by two anticommuting fields $\overline{\chi}$ and χ. The supersymmetry is made explicit by introducing anticommuting coordinates $(\theta, \overline{\theta})$ and a superfield $\Psi \equiv \psi + \frac{1}{2}(\overline{\theta}\chi + \theta\overline{\chi}) - \frac{1}{4}\overline{\theta}\theta\omega$, whence S may be written

$$S = \frac{1}{\Delta} \int \left(\frac{1}{2}\Psi(-\nabla_{SS}^2)\Psi + V(\Psi) \right) \mathrm{d}^d r \, \mathrm{d}\theta \, \mathrm{d}\overline{\theta} \tag{10}$$

where $\nabla_{SS}^2 = \nabla^2 + 4\partial^2/\partial\theta\partial\overline{\theta}$. This exhibits supersymmetry under rotations which leave $r^2 + \theta\overline{\theta}$ invariant. Parisi and Sourlas [6] argued that this theory exhibits a remarkable property of dimensionality reduction (for a nonperturbative proof see [10]): correlation functions whose arguments are restricted to a $(d-2)$-dimensional subspace are the same as those for a non-supersymmetric theory in $d-2$ dimensions, whose action is

$$S_{\text{Yang–Lee}} = \frac{1}{\Delta} \int \left(\frac{1}{2}\psi(-\nabla^2)\psi + V(\psi) \right) \mathrm{d}^{d-2} r \tag{11}$$

where in this case $V(\psi) = t\psi - \frac{1}{3}\psi^3$. There is one subtlety: before dropping the irrelevant terms, the contour in ψ should be rotated, in this case parallel to the imaginary axis, so as to make the integral defined nonperturbatively. The potential therefore becomes $\mathrm{i}t\psi + \frac{1}{3}\mathrm{i}\psi^3$. Thus (11) is just the action for the scaling theory of the Yang–Lee edge singularity [11], as

discussed by Fisher [12]. From (10) it is seen that Δ has dimension (length)$^{-2}$ and this is not affected by loop corrections, otherwise supersymmetry would be broken (this is presumably what happens in the random field Ising model, where dimensional reduction fails). It flows to infinity under the RG, and Δ^{-1} is irrelevant. However, it is a classic example of a *dangerously* irrelevant variable: it cannot be set equal to zero in the scaling theory. It is responsible for the modified hyperscaling relation $2 - \alpha_{\mathrm{BP}} = (d-2)\nu_{\mathrm{BP}}$.

Based on the above considerations, it is reasonable to conjecture that, up to possible constants, Δ and g should be identified, as should $x_{\mathrm{c}} - x$ and t (the above argument implies only that $t \propto x_0 - x$, but in fact $x_0 = x_{\mathrm{c}}$ for consistency with equation (1)). Thus, apart from non-universal constants, $G^{(r)}_{\mathrm{sing}}$ is given by the one-point function $\langle\Psi\rangle$ in the supersymmetric theory (10), which, by dimensional reduction, is the same as the one-point function $\langle\psi\rangle$ in the Yang–Lee scaling theory (11). For $d = 2$ the gradient terms are absent, so $G^{(r)}_{\mathrm{sing}}(x, g) = b_1(g\mathrm{d}/\mathrm{d}x)\ln Z_1$ where

$$Z_1 = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{(b_2/g)(-(x_{\mathrm{c}}-x)\psi + \frac{1}{3}\psi^3)}\,\mathrm{d}\psi$$

and b_1 and b_2 are non-universal constants. After rescaling the integration variable, this gives the main results (1), (2) of RBG, together with the values $\theta = \frac{1}{3}$, $\phi = \frac{2}{3}$ for the exponents.

According to (2), the scaling function $F(s)$ has singularities at the zeros of the Airy function, which lie on the negative real axis. The closest to $x = 0$ lies at $x_{\mathrm{c}}(g) = x_{\mathrm{c}} + (2.388\ldots)(g/b_2)^{2/3}$, governing the asymptotic behaviour $\sim x_{\mathrm{c}}(g)^{-N}$ of $\sum_A p^{(r)}_{N,A}\mathrm{e}^{-gA}$ as $N \to \infty$, for fixed small g. This singularity is a simple pole, corresponding to the value $\alpha_{\mathrm{BP}} \equiv 3 - \theta_{\mathrm{BP}} = 2$. All this agrees with general crossover theory [8] that the scaling function should exhibit the critical singularities of the stable fixed point.

We now discuss the generalization to area-weighted two-dimensional SALs at higher-order multicritical points. The additional interactions between nearby portions of the SAL will modify the parameters u_p in (7). As long as the truncation of terms leading to (9) remains valid, the dimensional reduction argument still applies with a modfied potential V, so that the rooted generating function is still given by the logarithmic derivative of a generalized Airy integral of the form $\int_C \mathrm{e}^{-V(\psi)/g}\,\mathrm{d}\psi$. The obvious candidates for potentials which then yield multicritical behaviour in the limit $g \to 0$ have the form $V(\psi) = \sum_{j=1}^{k} v_j\psi^j - \psi^{k+2}$ (the coefficient of ψ^{k+1} is redundant, as it can be removed by a shift in ψ). Here v_1 is linear in $x_{\mathrm{c}} - x$, and one can check that the other coefficients v_j should be positive deep inside the single-phase region. (Note that this form for V does not directly correspond to a multicritical point in the *branched polymer* ensemble when $g > 0$: the generalized Airy functions are still entire, so in general the branched polymer critical point will still come from its first zero, yielding $\alpha_{\mathrm{BP}} = 2$ as before. On the other hand, at the branched polymer collapse point it is believed [13] that $\alpha = 1$. This change of behaviour might be explained by a singularity in the Parisi–Sourlas mapping, rather than a critical point in V.)

Repeating the above analysis then leads to the result $G^{(r)}_{\mathrm{sing}}(v_j, g) = g(\mathrm{d}/\mathrm{d}v_1)\ln Z_k$ where

$$Z_k = \int_C \mathrm{e}^{-\left(\sum_{j=1}^{k} v_j\psi^j - \psi^{k+2}\right)/g}\,\mathrm{d}\psi \tag{12}$$

with the contour C chosen to guarantee convergence. Comparing with the scaling form (3) then gives the results (4), together with $\theta_k = 1/(k+2)$. In particular $\langle R^2\rangle \sim \langle N\rangle^{2\nu_k}$, where $\nu_k = 1/y_1(k) = (k+2)/2(k+1)$. It should be noted that, although these exponent values are based on extremizing a simple polynomial, they are not the same as those in Landau theory, in which the analogous potential would be $\sum_{j=1}^{k} v_j\phi^{2j} + \phi^{2(k+1)}$.

Finally, the limit $g \to 0$ may be taken in (12), using the saddle-point method, with the result that $G^{(r)}_{\rm sing}$ is simply given by a zero of $V'(\psi)$. By considering the limit when all the v_j are large and positive, it may be shown that the correct zero in this single-phase region is that on the real axis with the largest real part. The contour is to be run through this, locally parallel to the imaginary axis. Thus, near the $k = 2$ multicritical point, where $V(\psi) = v_1\psi + v_2\psi^2 - \psi^4$, one finds the result (5), as the appropriate root of a cubic equation. This formula has a number of interesting properties. At $v_2 = 0$, $G^{(r)}_{\rm sing}$ behaves like $(x_{\rm c} - x)^{1/3}$. For fixed $v_2 > 0$, the first singularity occurs not at $s = 1$, but at $s = -1$, and this is a square root: $G^{(r)}_{\rm sing} \sim (x_{\rm c}(v_2) - x)^{1/2}$, where $x_{\rm c}(v_2) - x_{\rm c}(0) \sim v_2^{3/2}$, all as expected on the basis of crossover theory. When $v_2 < 0$, the corresponding scaling function is $\Phi^<(s) = \left((s^2+1)^{1/2}+s\right)^{1/3} - \left((s^2+1)^{1/2}-s\right)^{1/3}$, where now $s = v_1/(-v_2)^{3/2}$. The other two roots remain complex, and there is now a first-order transition at $s \approx 1.6$, when Re V has the same value at each root. Note that $\Phi^<$, continued into the phase coexistence region, is analytic near the real axis and does not exhibit a spinodal singularity.

However, there is a puzzle, which our theory has in common with that of Saleur [7], associated with the physical identification of this multicritical point. The generalization of the Flory approximation [14] to the Θ and higher-order multicritical points, at which the first k renormalized virial coefficients vanish, may be phrased as follows: consider a long loop of length N and linear size R. In the absence of interactions, its entropy may be estimated on the basis of a free random walk to be $\mathrm{O}(R^2/N)$. The mean density is $\mathrm{O}(N/R^2)$, so that the interaction energy may be approximated by $v_{k+1}N(N/R^2)^k$, where v_{k+1} is the first non-vanishing virial coefficient. Balancing these two contributions to the free energy then gives $R \sim N^{\nu_k}$ with ν_k as above. This argument, combined with observation that at the kth multicritical point there should be exactly k relevant parameters, suggests that $k = 2$ in our theory should be identified with the Θ-point, and that, because of the simple scaling associated with the dimensionally reduced theory, the Flory result is in fact exact. However, the value $\nu_2 = \frac{2}{3}$, and the associated crossover exponent $y_2(2)/y_1(2) = \frac{2}{3}$, do not agree with the predictions of an exactly solvable model of Duplantier and Saleur [15], for which the corresponding values are $\frac{4}{7}$ and $\frac{3}{7}$. While it might be argued that this model is somewhat special, extensive numerical studies of more generic lattice models appear to confirm these values [16]. Even more strikingly, the supposedly correct value for ν at the Θ-point actually corresponds to $k = 6$ in our theory (and that of Saleur), and the crossover exponent corresponds to a perturbation with $j = 5$. But this relevant perturbation should lead to a multicritical point with $k = 3$, not the usual one with $k = 1$ as expected on the physical grounds for the Θ-point. This same paradox was present in Saleur's theory [7] and at present there seems to be no plausible resolution. It may well be that the coincidence of the Θ-point exponents with $k = 6$ is merely that, and that the sequence of multicritical points of SALs implied by Saleur's and the present theory represents some completely different physics. Unfortunately, because of the truncations made in going from the original SAL model to the generalized Airy integral, it is very difficult to say to what these other multicritical points might correspond physically.

To summarize, the conjectured scaling function of Richard *et al* [2] for area-weighted self-avoiding polygons has been shown to follow from physical reasoning concerning the crossover to branched polymers, together with the dimensional reduction argument of Parisi and Sourlas [6]. Depending on one's point of view, the numerical confirmation of this formula found by RGJ could be taken as dramatic vindication of the dimensional reduction argument, beyond its simple prediction of the value of the entropic exponent $\alpha_{\rm BP} = 2$. The exact formula for the scaling function is in accordance with standard crossover theory [8], but it points to the importance of understanding all the singularities of the crossover scaling function, not just the physical ones, in building up the full scaling form.

The formula proposed by RGJ is just the first of a series of exact scaling functions describing higher-order multicritical points for SALs weighted by their area. Moreover this approach enables one to recover exact results for scaling functions in the unweighted ensemble, and these have the form of algebraic functions. These are the first examples of exact but nontrivial scaling functions of more than one thermodynamic variable at isotropic critical points.

One might ask whether the present analysis throws any light on the central assumption of Richards *et al* [2] that the generating function satisfies a q-algebraic equation. Since the latter is essentially a lattice property, there appears to be no direct implication. One way to study this further would be to determine whether the forms of the corrections to scaling allowed by the field-theoretic approach are consistent with those implied by the existence of such an equation.

The simple structure found here is analogous to that which appears in $N = 2$ supersymmetric theories in two dimensions, although in this case the supersymmetry is of a different nature. From that point of view the multicritical points of SALs correspond to the A_{k+1} series of simple singularities [17] of the potential V: it would be interesting to find analogues of the D_{k+1} series, and the exceptional cases.

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After this Letter was written I was made aware of the important recent work of Brydges and Imbrie [18], in which they show that a particular model of branched polymers in D dimensions maps rigorously onto that of a hard-core gas (at negative fugacity) in $D - 2$ dimensions, and thence to a cut-off version of the field theory used in the present work, with a particular form for the potential V.

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[18] Brydges D C and Imbrie J Z 2001 *Preprint* math-ph/0107005